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Twisted Baryon Number in $N = 2$ Supersymmetric QCD

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Abstract

We show that the baryon number of $N = 2$ supersymmetric QCD can be twisted in order to couple the topological field theory of non-abelian monopoles to Spin^c -structures. To motivate the construction, we also consider some aspects of the twisting procedure as a gauging of global currents in two and four dimensions, in particular the rôle played by anomalies.

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1 Introduction

One of the most impressive consequences of the work by Seiberg and Witten [21, 22] has been the dual description of Donaldson theory in terms of monopole equations [29] (see [15, 18, 8, 24, 5] for a review). This has opened the way to new developments in four-dimensional topology and has provided astonishing new links between this field and the dynamics of supersymmetric gauge theories in four dimensions. One of these developments is the non-abelian generalization of the abelian Seiberg-Witten equations, known as non-abelian monopoles, which have been extensively studied both from the mathematical [19, 23, 20, 2] and the physical point of view [11, 12, 13, 9] in the last two years.

It is well-known that the Seiberg-Witten monopole equations involve Spin^c -structures on a four-manifold [14], and it was shown in [30] that these naturally arise when considering the duality transformations of the underlying physical theory. For the non-abelian case, Spin^c -structures were considered in [9] in order to define twisted $N = 2$ supersymmetric QCD on a general four-manifold. In this note, we will propose a different way to couple matter hypermultiplets to Spin^c -structures in non-abelian monopole theories. The idea is to consider an extended twisting procedure by gauging an additional global symmetry of the physical theory. The global symmetry turns out to be the baryon number $U(1)_B$. We will show in detail that the procedure is consistent and that the quantum numbers associated to this symmetry are the appropriate ones from the geometrical point of view. To motivate the construction, we will review some aspects of the twisting as a gauging procedure. A general conclusion of our analysis is that global symmetries not necessarily associated to the supersymmetry algebra could be considered in the construction of a topological field theory.

The organization of this paper is as follows: In section two we consider the twisting as a gauging in Donaldson-Witten theory and in two-dimensional topological sigma models. In section three we consider the gauging of the $U(1)_B$ current in $N = 2$ supersymmetric QCD and the coupling to Spin^c -structures. Finally, in section four we present our conclusions and outlook.

2 The twist as a gauging

Usually, the twisting procedure has been understood as a modification of the rotation group of a supersymmetric $N = 2$ theory through an embedding of the global symmetry associated to the supersymmetry algebra [26, 27, 10, 1]. An alternative (and equivalent) point of view is obtained when the twisting procedure is regarded as a gauging of an internal symmetry group, in which a global symmetry of the underlying supersymmetric model is promoted to a space-time symmetry. In many cases, the gauging is performed by adding to the Lagrangian of the original theory a new term, involving the coupling of the internal current to the Spin connection of the underlying manifold [7, 3, 4].

We will now discuss this approach to the twisting procedure in some detail in the case of $N = 2$ supersymmetric Yang-Mills, the model originally considered by Witten in [26]. The field content of the minimal $N = 2$ supersymmetric Yang-Mills theory with gauge group G on \mathbf{R}^4 is the following: a gauge field A_μ , two Majorana spinors $\lambda_{i\alpha}$, $i = 1, 2$, and their conjugates $\bar{\lambda}^{i\dot{\alpha}}$, a complex scalar B , and an auxiliary field D_{ij} (symmetric in i and j). The indices i, j denote the isospin indices of the internal symmetry group $SU(2)_I$ of $N = 2$ supersymmetry. The two Majorana spinors λ_1, λ_2 form a doublet of $SU(2)_I$. All these fields are considered in the adjoint representation of the gauge group G . The $SU(2)_I$ current of this model is given by:

$$j_a^\mu = \bar{\lambda} \sigma^\mu \sigma_a \lambda, \quad (2.1)$$

where $\sigma^\mu = (\mathbf{1}, i\sigma_a)$, and σ_a are the Pauli matrices.

If we try to formulate the theory on a general four-manifold X with Euclidean signature, the Majorana spinors $\lambda_{i\alpha}$ and their conjugates $\bar{\lambda}^{i\dot{\alpha}}$ are taken as independent Weyl spinors with opposite chirality, and we use the prescription of minimal coupling to gravity in the Lagrangian. The structure group of the spinor bundle is,

$$\text{Spin}_4 \simeq SU(2)_L \times SU(2)_R, \quad (2.2)$$

so that the covariant derivative acting on positive chirality spinors is given by,

$$D_\mu M_\alpha = \nabla_\mu M_\alpha - i\omega_\mu^a (\sigma_a)_\alpha{}^\beta M_\beta, \quad (2.3)$$

where ω_μ^a is the $SU(2)_L$ Spin connection on X , and ∇_μ is the covariant derivative on flat space (including the gauge connection). This construction can always be done locally, but globally we have of course the usual topological obstruction associated to the second Stiefel-Whitney class, $w_2(X)$. If X is not Spin, we cannot consistently couple the original $N = 2$ theory to gravity.

The twist of the theory under consideration involves the gauging of the $SU(2)_I$ group as the space-time symmetry group $SU(2)_L$. As the only fields charged with respect to $SU(2)_I$ are spinors, the gauging is achieved after adding to the Lagrangian the coupling of the $SU(2)_L$ connection to the $SU(2)_I$ current:

$$-\omega_\mu^a j_a^\mu = -\bar{\lambda}_\alpha^i (\sigma^\mu)^{\alpha\dot{\alpha}} (\sigma_a)_i{}^j \omega_\mu^a \lambda_{j\dot{\alpha}}. \quad (2.4)$$

The only change in the Lagrangian is in the fermion kinetic term, where the coupling to gravity becomes,

$$-\bar{\lambda}_\alpha^i (\sigma^\mu)^{\alpha\dot{\alpha}} \{ \delta_\alpha{}^\beta \delta_i{}^j - i \omega_\mu^a ((\sigma_a)_\alpha{}^\beta \delta_i{}^j + (\sigma_a)_i{}^j \delta_\alpha{}^\beta) \} \lambda_{j\dot{\alpha}}. \quad (2.5)$$

The connection appearing here is the tensor product connection on the bundle $S^+ \otimes S^+$, which is isomorphic to $\Omega_{\mathbb{C}}^0 \oplus \Omega_{\mathbb{C}}^{2,+}$. The scalar part corresponds to the antisymmetric part of $\lambda_{w\beta}$ while the self-dual two-form corresponds to the symmetric one. We can write (2.5) in terms of space-time fields, and to do this we introduce a scalar η , a one-form ψ_μ , and a self-dual two-form $\chi_{\mu\nu}$ as:

$$\begin{aligned} \eta &= -\lambda_\beta^\beta, & \psi_\mu &= i(\sigma_\mu)^{\dot{\alpha}\alpha} \bar{\lambda}_{\alpha\dot{\alpha}}, \\ \lambda_{(w\beta)} &= -\frac{1}{2\sqrt{2}} C^{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_\mu)_{w\dot{\alpha}} (\bar{\sigma}_\nu)_{\beta\dot{\beta}} \chi_{\mu\nu}, \end{aligned} \quad (2.6)$$

being $\bar{\sigma}^\mu = (\mathbf{1}, -i\sigma_a)$ on flat space (on a curved space, the usual vierbein is included). In terms of these fields the fermion kinetic terms can be written, after a lengthy computation, as

$$-\frac{i}{2} \psi^\mu D_\mu \eta - \sqrt{2} \psi_\mu D_\nu \chi^{\nu\mu}, \quad (2.7)$$

which are the standard fermion kinetic terms in Donaldson-Witten theory. Notice that the gauging of the $SU(2)_I$ global symmetry allows one to define the theory on an arbitrary smooth four-manifold, as the spinors become the differential forms in (2.6). Although these differential forms are complex, they have of course a natural underlying real structure. One must restrict

the resulting fields to be *real* differential forms, in order to have the same number of degrees of freedom in the untwisted and twisted theory. This counting of degrees of freedom must be taken into account if one is interested in extracting some information from the dynamics of the untwisted, physical theory.

In general, the gauging of a global symmetry can generate anomalies in the resulting theory. In the case of $N = 2$ supersymmetric Yang-Mills theory, the possible anomalies are related to the global $SU(2)$ anomaly discovered by Witten in [25], and it is easy to see [26] that they only appear when the corresponding moduli space is not orientable. This is not the case for the moduli space of ASD connections [6], which is the one described by Donaldson-Witten theory, and the twisted theory is then anomaly-free.

In the case of $N = 2$ supersymmetric Yang-Mills theory in four dimensions the twisting involves the gauging of a $SU(2)$ internal symmetry group. In the twisting of $N = 2$ supersymmetric sigma models in two dimensions, the rotation group is abelian, and the twisting involves now the global $SO(2) = U(1)$ rotation group. Because of that, the two-dimensional situation has a formal analogy with the twisting of the $U(1)_B$ current to be discussed in the next section, and we will examine it now in some detail.

From the point of view of the gauging procedure, it is better to start with an $N = 2$ supersymmetric sigma model involving chiral multiplets. The target manifold M is then Kähler and we have a bosonic field $\phi : \Sigma \longrightarrow M$ and a Dirac spinor $\psi_{\pm}^I \in \Gamma(\Sigma, S^{\pm} \otimes \phi^*(TM))$, where TM is the holomorphic tangent bundle to M . The kinetic fermion term in the action is:

$$S_f = \int_{\Sigma} d^2z i g_{I\bar{J}} (\psi_+^{\bar{I}} D_{\bar{z}} \psi_+^J + \psi_-^{\bar{I}} D_z \psi_-^J), \quad (2.8)$$

where the covariant derivatives are given in local coordinates by,

$$\begin{aligned} D_{\bar{z}} \psi_+^J &= \partial_{\bar{z}} \psi_+^J + \frac{i}{2} \omega_{\bar{z}} \psi_+^J + \Gamma_{K\bar{L}}^J \partial_{\bar{z}} \phi^K \psi_+^{\bar{L}}, \\ D_z \psi_-^J &= \partial_z \psi_-^J - \frac{i}{2} \omega_z \psi_-^J + \Gamma_{K\bar{L}}^J \partial_z \phi^K \psi_-^{\bar{L}}. \end{aligned} \quad (2.9)$$

This theory has a conserved, non-anomalous vector current $j_V^{\mu} = g_{I\bar{J}} \bar{\psi}^{\bar{I}} \gamma^{\mu} \psi^J$, with components,

$$j_V^z = 2g_{I\bar{J}} \psi_-^{\bar{I}} \psi_-^J, \quad j_V^{\bar{z}} = 2g_{I\bar{J}} \psi_+^{\bar{I}} \psi_+^J, \quad (2.10)$$

and an anomalous axial current $j_5^\mu = g_{IJ} \bar{\psi}^{\bar{I}} \gamma^\mu \gamma_5 \psi^J$ with components,

$$j_5^z = 2g_{IJ} \psi_-^{\bar{I}} \psi_-^J, \quad j_5^{\bar{z}} = -2g_{IJ} \psi_+^{\bar{I}} \psi_+^J. \quad (2.11)$$

The anomaly is given by the index of the Dirac operator and reads:

$$\int_\Sigma \phi^*(c_1(M)). \quad (2.12)$$

To twist the model we can gauge the $U(1)_V$ or the $U(1)_A$ symmetries. The first choice leads to the A model and the second one to the B model [27, 10, 28]. As in Donaldson-Witten theory, we promote the abelian global symmetry to a worldsheet space-time symmetry, and in this case this amounts to add to the Lagrangian the coupling of the corresponding currents to the worldsheet Spin connection. For the A model we have,

$$\begin{aligned} S_f &= \frac{i}{4} \int_\Sigma d^2z \sqrt{h} \omega_\mu j_V^\mu \\ &= \int_\Sigma d^2z \sqrt{h} i g_{IJ} \{ \psi_+^{\bar{I}} (\bar{\partial} \psi_+^J + \Gamma_{KL}^J \bar{\partial} \phi^K \psi_+^L) \\ &\quad + \psi_-^{\bar{I}} (\partial \psi_-^J - i \omega_z \psi_-^J + \Gamma_{KL}^J \partial \phi^K \psi_-^L) \}, \end{aligned} \quad (2.13)$$

and for the B model,

$$\begin{aligned} S_f &= \frac{i}{4} \int_\Sigma d^2z \sqrt{h} \omega_\mu j_A^\mu \\ &= \int_\Sigma d^2z \sqrt{h} i g_{IJ} \{ \psi_+^{\bar{I}} (\bar{\partial} \psi_+^J + i \omega_{\bar{z}} \psi_+^J + \Gamma_{KL}^J \partial_{\bar{z}} \phi^K \psi_+^L) \\ &\quad + \psi_-^{\bar{I}} (\partial \psi_-^J - i \omega_z \psi_-^J + \Gamma_{KL}^J \partial \phi^K \psi_-^L) \}. \end{aligned} \quad (2.14)$$

We see again that, in the twisted models, the fermion fields have changed their spin content. ψ_-^J becomes a $(0, 1)$ -form $\rho_{\bar{z}}^J$, while ψ_+^J becomes a scalar χ^J in the type A model, and a $(1, 0)$ -form ρ_z^J in the type B model.

It turns out [27] that the type A model can be formulated on any almost Hermitian target manifold. However, the type B model was obtained through the gauging of an anomalous current, and this can give ill-defined models: the anomaly in the global current, given in (2.12), is inherited in the twisted model as a global anomaly in the fermion determinant. This leads to additional restrictions on the geometry of the target space, as pointed out in

[28], because the fact that the $U(1)_A$ current is chiral leads to a non-linear sigma model anomaly. We will present here a computation of this global anomaly using the strategy of [17]. The fermion kinetic term of the type B model is:

$$S_B = \int_{\Sigma} d^2z \sqrt{h} g_{\bar{I}J} \{ \chi^{\bar{I}} D_z \rho_z^J + \theta^{\bar{I}} D_{\bar{z}} \rho_z^J \}, \quad (2.15)$$

where $\theta^{\bar{I}} = \psi_+^{\bar{I}}$ is a scalar in the twisted theory. The effective action is then a section of the line bundle,

$$\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2 = (\det D_z) \otimes (\det D_{\bar{z}}), \quad (2.16)$$

and the global, topological anomaly is measured by $c_1(\mathcal{L}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$. This can be computed using the index theorem for families as in [17]. Consider first the evaluation map:

$$\begin{aligned} \hat{\phi} : \text{Map}(\Sigma, M) \times \Sigma &\rightarrow M \\ (\phi, \sigma) &\mapsto \phi(\sigma). \end{aligned} \quad (2.17)$$

By pulling back differential forms on M through $\hat{\phi}^*$, we get differential forms on $\text{Map}(\Sigma, M) \times \Sigma$, with the natural bigrading given by the product structure:

$$\hat{\phi}^*(\omega) = \mathcal{O}_{\omega}^{(0)} + \mathcal{O}_{\omega}^{(1)} + \mathcal{O}_{\omega}^{(2)}, \quad (2.18)$$

where the $\mathcal{O}_{\omega}^{(i)}$ are of degree i with respect to Σ . This is precisely the descent procedure of [27]. The topological obstructions are given by:

$$c_1(\mathcal{L}_{1,2}) = \int_{\Sigma} \text{ch}(\hat{\phi}^*(\overline{TM})) \{ \pm 1 - \frac{1}{2} c_1(\Sigma) \}, \quad (2.19)$$

where we only keep the degree two forms on $\text{Map}(\Sigma, M)$. In $c_1(\mathcal{L})$ only the first descendant of $\hat{\phi}^*(\text{ch}(\overline{TM}))$ contributes, and we finally get,

$$c_1(\mathcal{L}) = \int_{\Sigma} c_1(\Sigma) \mathcal{O}_{c_1(M)}^{(0)}, \quad (2.20)$$

where $\mathcal{O}_{c_1(M)}^{(0)} = \hat{\phi}_{\sigma}^*(c_1(M))$ is the descendant of zero degree with respect to Σ , and $\hat{\phi}_{\sigma}$ is the map obtained from $\hat{\phi}$ by fixing a point $\sigma \in \Sigma$ (the cohomology class of the pulled-back form does not depend on the σ chosen). This result says then that the anomaly in the global current $U(1)_A$ is inherited in the twisted B model as a sigma model anomaly. The B model has no topological anomalies if the target is a Calabi-Yau manifold or if the worldsheet is a torus ($c_1(\Sigma) = 0$). The last possibility is natural, as in this case the twist does nothing (the torus is a hyperkähler manifold) and the original $N = 2$ supersymmetric model is anomaly-free.

3 Twisting the $U(1)_B$ current in $N = 2$ supersymmetric QCD

To be specific, we will consider $N = 2$ supersymmetric QCD with gauge group $SU(N)$ and $N_f = 1$ in the fundamental representation (we will follow the notations and conventions in [12, 16]). In this case there is one $N = 2$ matter hypermultiplet which contains a complex scalar isodoublet $q = (q_1, q_2)$, fermions $\psi_{q\alpha}, \psi_{\tilde{q}\alpha}, \bar{\psi}_{q\dot{\alpha}}, \bar{\psi}_{\tilde{q}\dot{\alpha}}$, and a complex scalar isodoublet auxiliary field F_i . The fields $q_i, \psi_{q\alpha}, \bar{\psi}_{\tilde{q}\dot{\alpha}}$, and F_i are in the fundamental representation of the gauge group, while the fields $q^{i\dagger}, \psi_{\tilde{q}\alpha}, \bar{\psi}_{q\dot{\alpha}}$, and $F^{i\dagger}$ are in the conjugate representation. From the point of view of $N = 1$ superspace, this multiplet contains two $N = 1$ chiral multiplets and therefore it can be described by two $N = 1$ chiral superfields Q and \tilde{Q} , *i.e.*, these superfields satisfy the constraints $\bar{D}_{\dot{\alpha}}Q = 0$ and $\bar{D}_{\dot{\alpha}}\tilde{Q} = 0$. While the superfield Q is in the fundamental representation of the gauge group, the superfield \tilde{Q} is in the corresponding conjugate representation. The component fields of these $N = 1$ superfields are:

$$\begin{aligned} Q, Q^\dagger &\longrightarrow q_1, \psi_{q\alpha}, F_2, q^{1\dagger}, \bar{\psi}_{q\dot{\alpha}}, F^{2\dagger}, \\ \tilde{Q}, \tilde{Q}^\dagger &\longrightarrow q^{2\dagger}, \psi_{\tilde{q}\alpha}, F^{1\dagger}, q_2, \bar{\psi}_{\tilde{q}\dot{\alpha}}, F_1. \end{aligned} \quad (3.21)$$

The $SU(2)_I$ current includes now a contribution from the bosonic part of the hypermultiplet:

$$j_a^\mu = \bar{\lambda}\sigma^\mu\sigma_a\lambda - iq^\dagger\sigma_a D^\mu q + iD^\mu q^\dagger\sigma_a q. \quad (3.22)$$

The twist of the theory can also be understood as a gauging of the $SU(2)_I$ current, as it happens with the pure $N = 2$ supersymmetric Yang-Mills theory. This is achieved by adding to the original Lagrangian a term $-\omega_\mu^a j_a^\mu + q^\dagger \omega_\mu^a \omega^{\mu b} \sigma_a \sigma_b q$. The kinetic term for the bosons in the resulting theory is then,

$$(D_\mu + i\omega_\mu^a \sigma_a)q^\dagger (D_\mu - i\omega^{\mu a} \sigma_a)q, \quad (3.23)$$

where D_μ is the covariant derivative acting on scalars in the fundamental of $SU(N)$. We then see that, after the twisting, the bosonic fields q become positive-chirality spinors. The fields in the $N = 2$ hypermultiplet are

redefined after the twisting as follows:

$$\begin{aligned}
q_i &\longrightarrow M_\alpha, \\
\psi_{q\alpha} &\longrightarrow -\mu_\alpha/\sqrt{2}, \\
\bar{\psi}_{\bar{q}}^{\dot{\alpha}} &\longrightarrow v^{\dot{\alpha}}/\sqrt{2}, \\
q^{\dagger i} &\longrightarrow \bar{M}^\alpha, \\
\bar{\psi}_{q\dot{\alpha}} &\longrightarrow \bar{v}_{\dot{\alpha}}/\sqrt{2}, \\
\psi_{\bar{q}}^\alpha &\longrightarrow \bar{\mu}^\alpha/\sqrt{2}.
\end{aligned}
\tag{3.24}$$

The motivation for this redefinition is geometrical and can be understood using the Mathai-Quillen formulation of the twisted theory [11]: the fields $\mu_\alpha, \bar{\mu}^\alpha$ are a basis of differential forms for the configuration space of monopole fields M_α, \bar{M}^α , and the fields $v^{\dot{\alpha}}, \bar{v}_{\dot{\alpha}}$ are a basis of fields for the fibre where the moduli equations take values.

We have seen that, after the twisting, the bosonic fields q in the hypermultiplet become positive-chirality spinors: the gauging of the $SU(2)_I$ current makes possible to define $N = 2$ supersymmetric Yang-Mills theory on a curved manifold, but the obstruction associated to $w_2(X)$ reappears when matter hypermultiplets are introduced. From the point of view of four-dimensional geometry it would be desirable to construct twisted $N = 2$ supersymmetric QCD on a general four-manifold. In the case of the Seiberg-Witten monopole equations, the issue is precisely to consider Spin^c -structures, and one would like to extend this possibility to the non-abelian generalization of these equations. The idea is to construct an extended twisting procedure by gauging an additional global symmetry of the physical theory. As the pure Yang-Mills sector is already well-defined with the usual gauging of the $SU(2)_I$ isospin group, this symmetry can only act on the matter sector. Moreover, Spin^c -structures involve a $U(1)$ gauge group associated to a line bundle L over the four-manifold X . In fact, in four dimensions we have that,

$$\text{Spin}_4^c = \{(A, B) \in U(2) \times U(2) : \det(A) = \det(B)\}, \tag{3.25}$$

and therefore the structure groups of the complex spinor bundles $S^\pm \otimes L^{1/2}$ are $SU(2)_{L,R} \times U(1)$, with the same $U(1)$ action in both sectors. We should then gauge a global, non-anomalous $U(1)$ symmetry in the original $N = 2$

theory, acting solely on the matter hypermultiplets. The required symmetry is precisely the baryon number. Let us analyze this in some detail.

The global anomaly-free symmetry of $N = 2$ supersymmetric QCD with N_f hypermultiplets is:

$$SU(N_f) \times U(1)_B \times SU(2)_I. \quad (3.26)$$

For $N_f = 1$, the baryon number $U(1)_B$ acts on the hypermultiplet as,

$$\begin{aligned} Q &\rightarrow e^{i\phi} Q, & \tilde{Q} &\rightarrow e^{-i\phi} \tilde{Q}, \\ Q^\dagger &\rightarrow e^{-i\phi} Q, & \tilde{Q}^\dagger &\rightarrow e^{i\phi} \tilde{Q}^\dagger. \end{aligned} \quad (3.27)$$

As it is a vector symmetry, it is non-anomalous. In components it reads:

$$\begin{aligned} q &\rightarrow e^{i\phi} q, & q^\dagger &\rightarrow e^{-i\phi} q^\dagger, \\ \psi_{q\alpha} &\rightarrow e^{i\phi} \psi_{q\alpha}, & \bar{\psi}_{\tilde{q}\dot{\alpha}} &\rightarrow e^{i\phi} \bar{\psi}_{\tilde{q}\dot{\alpha}}, \\ \bar{\psi}_{q\dot{\alpha}} &\rightarrow e^{-i\phi} \bar{\psi}_{q\dot{\alpha}}, & \psi_{\tilde{q}\alpha} &\rightarrow e^{-i\phi} \psi_{\tilde{q}\alpha}. \end{aligned} \quad (3.28)$$

The $U(1)_B$ current associated to this symmetry is:

$$j_B^\mu = -iD_\mu q^\dagger q + iq^\dagger D^\mu q + \bar{\psi}_{q\dot{\alpha}} (\sigma^\mu)^{\dot{\alpha}\alpha} \psi_{q\alpha} - \bar{\psi}_{\tilde{q}\dot{\alpha}} (\sigma^\mu)^{\dot{\alpha}\alpha} \psi_{\tilde{q}\alpha}. \quad (3.29)$$

To gauge this $U(1)$ symmetry, consider the determinant line bundle L associated to a Spin^c -structure on X , endowed with a connection b_μ , and add to the Lagrangian the term:

$$\frac{1}{2} j_B^\mu b_\mu - \frac{1}{4} q^\dagger b_\mu b^\mu q. \quad (3.30)$$

If we gauge both the $SU(2)_I$ and the $U(1)_B$ symmetries, the covariant derivatives acting on the components of the matter hypermultiplet in the resulting Lagrangian are the appropriate ones for complex spinors taking values in $S^\pm \otimes L^{1/2} \otimes E$. To further analyze the consistency of the procedure, it is useful to consider the correspondence between the fields in the original $N = 2$ theory and the fields appearing in the Mathai-Quillen formulation of the moduli problem. This correspondence is given in (3.24), and the structure of (3.30) implies that the components of the matter hypermultiplet will be sections of $L^{\pm\frac{1}{2}}$ if their baryon number is ± 1 . Taking into account the baryon number assignment in (3.28), we see that the fields M_α, μ_α are sections of

$S^+ \otimes L^{1/2} \otimes E$, $v^{\dot{\alpha}}$ is a section of $S^- \otimes L^{1/2} \otimes E$, \overline{M}^α , $\bar{\mu}^\alpha$ are sections of $S^+ \otimes L^{-1/2} \otimes \overline{E}$, and $\bar{v}_{\dot{\alpha}}$ is a section of $S^- \otimes L^{-1/2} \otimes \overline{E}$. This is then consistent with the expected structure of the complex spinor bundles, with the same determinant line bundle for both S^\pm . The kinetic terms in the twisted theory are then,

$$\mathcal{L}_k = g_{\mu\nu} D_L^\mu \overline{M}^\alpha D_L^\nu M_\alpha - \frac{i}{2} (\bar{v}_{\dot{\alpha}} D_L^{\dot{\alpha}\alpha} \mu_\alpha + \bar{\mu}^\alpha D_{L\alpha\dot{\alpha}} v^{\dot{\alpha}}), \quad (3.31)$$

where,

$$D_L^\mu = D_\mu + \frac{i}{2} b_\mu, \quad (3.32)$$

is the covariant derivative associated to the tensor product connection on $S^+ \otimes L^{1/2} \otimes E$, D_μ is given in (2.3), and $D_L^{\dot{\alpha}\alpha}$ is the corresponding Dirac operator. The remaining terms in the Lagrangian are the same as in the non-abelian monopole theory constructed in [11].

As in the case of the usual twisting involving matter hypermultiplets [13, 16], the Lagrangian obtained after the gauging of the $SU(2)_I$ and the $U(1)_B$ symmetries is not Q -closed. This is a general feature of the twisting procedure: in order to guarantee the existence of a scalar topological symmetry on a curved manifold it can be necessary to add to the Lagrangian terms involving the non-trivial geometry of this manifold. If we compute $[Q, \mathcal{L}]$, we must use the Weitzenböck formula for the Spin^c -case [14]:

$$D_{L\alpha\dot{\alpha}}^\dagger D_L^{\dot{\alpha}\beta} = (D_{L\mu} D_L^\mu + \frac{1}{4} R) \delta_\alpha^\beta + i F_\alpha^\beta + \frac{i}{2} \Omega_\alpha^\beta, \quad (3.33)$$

where R is the scalar curvature of the manifold, and F_α^β and Ω_α^β are the self-dual part of the gauge field strength on E and the curvature of the line bundle L , respectively. The kinetic terms in (3.31), \mathcal{L}_k , give a non-zero contribution:

$$[Q, \mathcal{L}_k] = -\frac{R}{4} (\overline{M}^\alpha \mu_\alpha + \bar{\mu}^\alpha M_\alpha) - \frac{i}{2} (\overline{M}^\alpha \Omega_\alpha^\beta \mu_\beta + \bar{\mu}^\alpha \Omega_\alpha^\beta M_\alpha). \quad (3.34)$$

It then follows that the modified Lagrangian,

$$\mathcal{L}_{\text{top}} = \mathcal{L} + \frac{R}{4} \overline{M}^\alpha M_\alpha + \frac{i}{2} \overline{M}^\alpha \Omega_\alpha^\beta M_\alpha, \quad (3.35)$$

where \mathcal{L} contains the kinetic term \mathcal{L}_k in (3.31) plus the rest of the terms of the theory of non-abelian monopoles [11], is Q -closed on a general four-manifold.

This Lagrangian was obtained in [9], and a standard analysis shows that the resulting topological field theory corresponds to the moduli problem encoded in the equations

$$\begin{aligned} F_{\alpha\beta}^{+a} + i\overline{M}_{(\alpha}(T^a)M_{\beta)} &= 0, \\ D_L^{\alpha\dot{\alpha}}M_{\alpha} &= 0. \end{aligned} \quad (3.36)$$

These are equations for a pair (A, M) consisting of a connection A on E and a section M of the complex spinor bundle $S^+ \otimes L^{1/2} \otimes E$, where the connection on the determinant line bundle L is fixed. The operator D_L is just the Dirac operator for this twisted bundle. Similar equations have been considered in the mathematical literature, see [19, 2, 8, 20]. Notice that the twist of the $U(1)_B$ current gives precisely the geometrical content of these moduli equations. As it was expected, the resulting topological field theory is anomaly free, because the determinant line bundle of the twisted Dirac operator D_L is always orientable due to its underlying complex structure.

Usually, the fact that the theory is topological means that correlation functions do not depend on the Riemannian metric of the four-manifold. In the same way one can easily check that the theory is topological with respect to the Spin^c -connection: the correlation functions do not depend on the choice of the connection b_μ on the line bundle L , but only on the topological class of the Spin^c -structure. To see this, notice that the Mathai-Quillen formulation of the model coupled to a Spin^c -structure is almost identical to the one presented in [11], the only difference being that we must consider instead the expression for the Dirac operator D_L (including the connection b_μ on the determinant line bundle L). The Q -transformations of the fields in this off-shell formulation are identical to the ones in the Spin case and do not depend on the connection on L . As the full Lagrangian (3.35) is Q -exact, we have, using the results in [11]:

$$\frac{\delta \mathcal{L}_{\text{top}}}{\delta b_\mu} = \{Q, \frac{1}{2}(\bar{v}_{\dot{\alpha}}(\sigma^\mu)^{\dot{\alpha}\alpha}M_\alpha - \overline{M}^\alpha(\sigma^\mu)_{\dot{\alpha}\alpha}v^{\dot{\alpha}})\}. \quad (3.37)$$

This in turn guarantees that the twisted theory is independent of the choice of b_μ .

Notice that the metric (or Spin connection) and the Spin^c -connection enter the construction on the same footing. In this formulation of non-abelian monopoles coupled to Spin^c -structures, the connection on the determinant

line bundle L is a background gauge field just like the Spin connection. The analysis of the moduli problem associated to the equations (3.36) is very similar to the one presented in [11] in the Spin case, but one should not divide by the group of gauge transformations associated to the $U(1)$ gauge field, because this is a *background* field. This is in contrast with the corresponding situation in the abelian theory. However, the topological correlation functions depend on the topological class of the Spin^c -structure chosen to gauge the $U(1)_B$ symmetry. In particular, the virtual dimension of the moduli space depends now on the first Chern class of L , and other features (like the orientability, the analysis of reducibles and the structure of the observables) are almost identical to the Spin case.

4 Conclusions and outlook

We have shown that $N = 2$ supersymmetric QCD has the possibility of coupling the matter fields to Spin^c -structures once the adequate symmetry has been identified. The gauging of the $U(1)_B$ current can be generalized to theories with more than one hypermultiplet: in this case there are N_f $U(1)$ symmetries that can be gauged, and this makes possible to consider N_f different Spin^c -structures, as it has been already noticed in [9]. An obvious extension of this work would be the computation, using physical methods, of the topological correlation functions of this extended model, generalizing in this way the results in [12]. One should be careful with the subtleties involving Spin^c -structures that have been discussed in [30]. But the moral of our procedure is perhaps that in the construction of topological quantum field theories one could consider not only the symmetries associated to the supersymmetry algebra, but *any* global symmetry of the theory which might have a geometrical meaning through an appropriate gauging.

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